

A FAST FLATNESS TESTING ALGORITHM IN CHARACTERISTIC ZERO

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ABSTRACT. We prove a fast computable criterion that expresses non-flatness in terms of torsion: Let R be a regular algebra of finite type over a field \underline{k} of characteristic zero and let F be a module finitely generated over an R -algebra of finite type. Given a maximal ideal \mathfrak{m} in R , let S be the coordinate ring of the blowing-up of $\text{Spec } R$ at the closed point \mathfrak{m} . Then $F_{\mathfrak{m}}$ is flat over $R_{\mathfrak{m}}$ if and only if $F_{\mathfrak{m}} \otimes_R S$ is a torsion-free $R_{\mathfrak{m}}$ -module. If $\underline{k} = \mathbb{R}$ or \mathbb{C} , we give a stronger criterion – without the regularity assumption on R . We also show the corresponding results in the real- and complex-analytic categories.

1. INTRODUCTION

Flatness of a morphism $\varphi : X \rightarrow Y$ of algebraic or analytic varieties is a fundamental property, which – when present – allows one to regard the fibres of φ as a family of varieties parametrized by a given variety Y . It is therefore interesting to know how to verify whether or not a given morphism is flat. However, determining flatness is, in general, a difficult task. The purpose of the present paper is to give criteria that express flatness in terms of torsion-freeness and are easily computable using computer algebra.

Our flatness criteria (see Section 1.1, below) assert that flatness of φ at a point $\xi \in X$ can be detected as follows: Let $\eta = \varphi(\xi) \in Y$ and let $\sigma : Z \rightarrow Y$ be the blowing-up of Y at η , with $\zeta \in \sigma^{-1}(\eta)$. Then φ is flat at ξ if and only if the pull-back of φ by σ , $X \times_Y Z \rightarrow Z$ has no torsion at (ξ, ζ) ; i.e., the local ring $\mathcal{O}_{X \times_Y Z, (\xi, \zeta)}$ is a torsion-free $\mathcal{O}_{Z, \eta}$ -module. The simplicity of the latter condition, from the computational point of view, is best seen in the following result.

Theorem 1.1. *Let \underline{k} be a field of characteristic zero and let $R = \underline{k}[y_1, \dots, y_n]$. Let F be a module finitely generated over an R -algebra of finite type, say, $F \cong R[x]^q/M$, where $x = (x_1, \dots, x_m)$, $m \geq 1$, and M is a submodule in $R[x]^q$. Set $\widetilde{M} = M(y_1 y_n, \dots, y_{n-1} y_n, y_n, x)$, i.e., let \widetilde{M} be the module obtained from M by substituting $y_j y_n$ for y_j , $j = 1, \dots, n-1$. Then $F_{(x, y)}$ is a flat $R_{(y)}$ -module if and only if $\widetilde{M} = \widetilde{M} : y_n$ (as $R[x]$ -submodules of $R[x]^q$).*

The idea of expressing flatness in terms of zerodivisors dates back to Auslander's seminal paper [5]. Auslander showed that flatness of a finitely generated module over a regular local ring is equivalent to torsion-freeness of a sufficiently high tensor power of the module. In recent years, his theorem was extended to modules finite over an essentially finite-type morphism of schemes (or a holomorphic mapping of

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complex-analytic spaces): by Adamus, Bierstone and Milman [2] in the analytic and complex-algebraic categories (extending a special case done by Galligo and Kwieciński [11]), and by Avramov and Iyengar [6] in the category of schemes smooth over a field.

All of the above three generalizations follow the philosophy of Auslander’s proof, and consequently all share the same limitations for practical application. First of all, they require the base ring to be regular (or even smooth, in [6]). This in itself is not yet the most restrictive assumption. As the authors show in [4], it is not difficult to generalize to the singular case (at least in the complex-analytic and complex-algebraic categories). More importantly, in general, the above criteria detect non-flatness of a module by finding torsion only in its n -fold tensor power, where n is the Krull dimension of the base ring. In fact, this is the case already for finite modules: Auslander [5] shows an example of a non-flat module F finitely generated over a regular local ring R of dimension n such that F as well as all its tensor powers up to $(n-1)$ ’st are torsion-free over R . To put this in a perspective of actual calculations, consider the module from Example 5.2, below: There we have $R = \mathbb{C}[y_1, y_2, y_3]$ and a non-flat R -module F finitely generated over $R[x_1, \dots, x_9]$. To verify the non-flatness of F by means of [11], [2] or [6], one would need to perform primary decomposition (cf. [2, Rem. 1.4]) of an ideal in $3 + 3 \cdot 9 = 30$ variables! This is, of course, practically impossible.

Let us consider for a moment the geometric point of view. Let $\varphi : X \rightarrow Y$ denote a morphism of algebraic or analytic varieties, with Y smooth of dimension n , and $\varphi(\xi) = \eta$. The general philosophy of [2] and [6] is that the non-flatness of φ at the point ξ means that the fibre $\varphi^{-1}(\eta)$ is somehow bigger than the generic fibre of φ . Passing to fibred powers of φ (which corresponds to taking tensor powers of the local ring $\mathcal{O}_{X,\xi}$ of the source over $\mathcal{O}_{Y,\eta}$) amplifies the difference between the special and the generic fibre of the mapping to the extent that in the n -fold fibred power the special fibres themselves form an irreducible component of the source. (This component is responsible for the $\mathcal{O}_{Y,\eta}$ -torsion in the n -fold tensor power of $\mathcal{O}_{X,\xi}$.)

Here we take a different approach. The main idea behind our results is that the “bigness” of the fibre $\varphi^{-1}(\eta)$ can be amplified much quicker. Namely, by taking fibred product with a morphism $\sigma : Z \rightarrow Y$ with generically finite fibres, whose fibre over η is of codimension 1 in Z (cf. Theorems 1.5 and 1.8, below).

The main tool of this paper is Hironaka’s criterion for flatness (Theorem 2.1, below). Consequently, we first establish our criteria in the analytic category. We then derive the corresponding results in the algebraic setting by standard faithful flatness arguments.

1.1. Main results. Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Our main results are the following two flatness criteria. In fact, Theorem 1.2 is a special case of Theorem 1.3, but we choose to state and prove it separately, because this special case is already quite important and its proof allows one to better understand the somehow technical proof of the latter criterion.

Theorem 1.2. *Let F be an analytic module over $R = \mathbb{K}\{y_1, \dots, y_n\}$. Let $S = \mathbb{K}\{z_1, \dots, z_n\}$ and let $\kappa : R \rightarrow S$ be the morphism defined as*

$$\kappa(y_1) = z_1 z_n, \dots, \kappa(y_{n-1}) = z_{n-1} z_n, \kappa(y_n) = z_n.$$

Then F is a flat R -module if and only if y_n is not a zerodivisor on $F \tilde{\otimes}_R S$ (as an S -, or equivalently, as an R -module).

Here, $\mathbb{K}\{y\}$ denotes the ring of convergent power series with coefficients in \mathbb{K} . An *analytic module* over R means (after [12]) a module which is finitely generated over some local analytic R -algebra, i.e., a ring of the form $R\{x\}/J$, where $x = (x_1, \dots, x_m)$ and J is an ideal in $R\{x\}$ ¹. The analytic tensor product, denoted $\tilde{\otimes}_R$, is simply the coproduct in the category of local analytic R -algebras (see, e.g., [2]).

Below, we denote by $\text{Specan}(\mathbb{K}\{y\}/I)$ the germ (at the origin) of a \mathbb{K} -analytic space defined by an ideal I in $\mathbb{K}\{y\}$.

Theorem 1.3. *Let I be an ideal in $\mathbb{K}\{y_1, \dots, y_n\}$, let $R = \mathbb{K}\{y_1, \dots, y_n\}/I$, and assume that $\text{Specan}(R)$ is positive-dimensional. Let F be an analytic module over R . Let $S = \mathbb{K}\{z_1, \dots, z_n\}$ and $\kappa : \mathbb{K}\{y_1, \dots, y_n\} \rightarrow S$ be as in Theorem 1.2, and let I^* be the strict-transform ideal of I ; i.e.,*

$$I^* = \{g \in S : z_n^k \cdot g \in I \cdot S \text{ for some } k \in \mathbb{N}\}.$$

Suppose that I^ is a proper ideal in S . Then:*

- (i) *If F is not R -flat, then y_n is a zerodivisor on $F \tilde{\otimes}_R S/I^*$.*
- (ii) *If S/I^* is an integral domain (and R is a subring of S/I^*), then R -flatness of F implies that $F \tilde{\otimes}_R S/I^*$ is torsion-free over S/I^* (resp. over R).*

Remark 1.4. The assumption that I^* is a proper ideal in S is equivalent to the assumption on the strict transform in Theorem 1.8, below. Therefore it is always satisfied, after a linear change in the y -variables if needed (cf. Remark 1.9).

To formulate the geometric analogues of the above results, we need to introduce the notion of vertical component. Let $\varphi_\xi : X_\xi \rightarrow Y_\eta$ denote a morphism of germs of \mathbb{K} -analytic spaces, and let W_ξ denote an irreducible component of X_ξ (isolated or embedded). Recall ([2]) that W_ξ is called an *algebraic* (respectively, *geometric*) *vertical component* of φ_ξ (or over Y_η) if φ_ξ maps W_ξ to a proper analytic (respectively, nowhere-dense) subgerm of Y_η . (More precisely, for a sufficiently small representative W of W_ξ , and a corresponding representative φ of φ_ξ , the germ $\varphi(W)_\eta$ is a proper analytic (respectively, nowhere-dense) subgerm of Y_η .) A component of X_ξ is isolated (resp. embedded) if its defining prime ideal in the local ring $\mathcal{O}_{X,\xi}$ is an isolated (resp. embedded) prime. For the basic facts and terminology on blowing-up and strict transforms, we refer the reader to [15].

Theorem 1.5. *Let $\varphi : X \rightarrow Y$ be a morphism of \mathbb{K} -analytic spaces, with Y smooth at a point η . Let $\sigma : Z \rightarrow Y$ denote the blowing-up of $\eta \in Y$ restricted to a local coordinate chart, say, centered at $\zeta \in Z$. Let $\xi \in X$ be a point such that $\varphi(\xi) = \eta$, and let F be a finitely generated $\mathcal{O}_{X,\xi}$ -module. Then F is a flat $\mathcal{O}_{Y,\eta}$ -module if and only if $F \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{Z,\zeta}$ is torsion-free over $\mathcal{O}_{Z,\zeta}$ (or, equivalently, over $\mathcal{O}_{Y,\eta}$).*

Corollary 1.6. *Let $\varphi : X \rightarrow Y$ be a morphism of complex-analytic spaces, with Y smooth at a point η . Let $\sigma : Z \rightarrow Y$ denote the blowing-up of $\eta \in Y$ restricted to a local coordinate chart, say, centered at $\zeta \in Z$. Let $\xi \in X$ be a point such that $\varphi(\xi) = \eta$. Then the map germ φ_ξ is flat if and only if its pull-back by σ_ζ , $(X \times_Y Z)_{(\xi,\zeta)} \rightarrow Z_\zeta$ has no algebraic vertical components. (Equivalently, the induced map germ $(X \times_Y Z)_{(\xi,\zeta)} \rightarrow Y_\eta$ has no algebraic vertical components.)*

¹Such a module is also called an *almost finitely generated* R -module; see, e.g., [11] or [2].

Remark 1.7. Corollary 1.6 follows from Theorem 1.5 applied to $F = \mathcal{O}_{X,\xi}$. Indeed, there is an isomorphism $\mathcal{O}_{X,\xi} \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{Z,\zeta} \cong \mathcal{O}_{X \times_Y Z, (\xi, \zeta)}$, and every associated prime $\mathfrak{q} \in \mathcal{O}_{Y,\eta}$ of the $\mathcal{O}_{Y,\eta}$ -module $\mathcal{O}_{X \times_Y Z, (\xi, \zeta)}$ is of the form $\mathfrak{q} = \mathfrak{p} \cap \mathcal{O}_{Y,\eta}$ for some associated prime \mathfrak{p} of the ring $\mathcal{O}_{X \times_Y Z, (\xi, \zeta)}$. Therefore, $\mathcal{O}_{X,\xi} \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{Z,\zeta}$ has a zerodivisor over $\mathcal{O}_{Y,\eta}$ if and only if some associated prime of $\mathcal{O}_{X \times_Y Z, (\xi, \zeta)}$ contracts to a non-zero ideal in $\mathcal{O}_{Y,\eta}$. In other words, the corresponding irreducible component of $X \times_Y Z$ through (ξ, ζ) is mapped to a proper analytic subgerm of Y_η .

It is interesting to compare the above result with Theorem 1.1 of [2], where the non-flatness of φ_ξ is detected in the n -fold fibred power $\varphi_{\xi\{n\}}^{\{n\}}$. There, the characterisation is in terms of the *geometric* vertical components, and it is actually an open problem ([2, Question 1.11]) whether it can be stated in terms of the algebraic vertical components as well. Clearly, every geometric vertical component over an irreducible target is algebraic vertical, but the converse is not true, in general (see, e.g., [1]).

Theorem 1.5 is, in fact, a special case of the following result, which is a geometric analogue of Theorem 1.3.

Theorem 1.8. *Let $\varphi : X \rightarrow Y$ be a \mathbb{K} -analytic mapping of \mathbb{K} -analytic subspaces of \mathbb{K}^m and \mathbb{K}^n respectively. Suppose that $0 \in X$, $\varphi(0) = 0 \in Y$, and Y is positive-dimensional at 0. Let $\sigma : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be the blowing-up of the origin restricted to a local coordinate chart in which the strict transform Z of Y passes through the origin. Let F be a finitely generated $\mathcal{O}_{X,0}$ -module. Then:*

- (i) *If F is not flat over $\mathcal{O}_{Y,0}$, then $y_n \in \mathcal{O}_{Y,0}$ is a zerodivisor on $F \tilde{\otimes}_{\mathcal{O}_{Y,0}} \mathcal{O}_{Z,0}$.*
- (ii) *If the germ Z_0 is irreducible and $(\sigma|_Z)_0^* : \mathcal{O}_{Y,0} \rightarrow \mathcal{O}_{Z,0}$ is a monomorphism, then $\mathcal{O}_{Y,0}$ -flatness of F implies that $F \tilde{\otimes}_{\mathcal{O}_{Y,0}} \mathcal{O}_{Z,0}$ is torsion-free over $\mathcal{O}_{Z,0}$ (and over $\mathcal{O}_{Y,0}$).*

Like Theorem 1.5, naturally, the above statement also has its version for flatness of φ at the origin in terms of vertical components, analogous to Corollary 1.6.

Remark 1.9. Notice that there always exists a local coordinate chart of the blowing-up of the origin in which the strict transform of Y passes through the origin (of that chart), because, by assumption, $0 \in Y$ is nowhere-dense in Y . More precisely, after a linear change in the y -variables if needed, one can assume that the vector $(0, \dots, 0, 1) \in \mathbb{K}^n$ belongs to the tangent cone of Y at the origin. Then the strict transform of Y passes through the origin in the local coordinate chart for which y_n is the exceptional divisor.

Proof of Theorem 1.8. Let $y = (y_1, \dots, y_n)$, $z = (z_1, \dots, z_n)$, and let I be an ideal in $\mathbb{K}\{y\}$ such that $\mathcal{O}_{Y,0} = \mathbb{K}\{y\}/I$. Set $R = \mathcal{O}_{Y,0}$, $S = \mathbb{K}\{z\}$, and define $\kappa : \mathbb{K}\{y\} \rightarrow S$ as $\kappa = \sigma_0^*$. Then $\mathcal{O}_{Z,0} \cong S/I^*$, where I^* is the ideal from Theorem 1.3, and the assumption that Z passes through the origin implies that I^* is a proper ideal in S . Clearly, F is an analytic module over R . Therefore, our assertions follow directly from Theorem 1.3. \square

Finally, let us state the algebraic analogue of (the exciting part of) Theorem 1.3. Here, as before, $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .

Theorem 1.10. *Let $R = \mathbb{K}[y_1, \dots, y_n]/I$, where I is a proper ideal in $\mathbb{K}[y_1, \dots, y_n]$. Let $A = R[x_1, \dots, x_m]/Q$ be an R -algebra of finite type, and let F be a finitely*

generated A -module. Let $S = \mathbb{K}[z_1, \dots, z_n]$ and let $\kappa : \mathbb{K}[y_1, \dots, y_n] \rightarrow S$ be the morphism defined as

$$\kappa(y_1) = z_1 z_n, \dots, \kappa(y_{n-1}) = z_{n-1} z_n, \kappa(y_n) = z_n.$$

Let I^* be the strict-transform ideal of I ; i.e.,

$$I^* = \{g \in S : z_n^k \cdot g \in I \cdot S \text{ for some } k \in \mathbb{N}\}.$$

Suppose that I^* is a proper ideal in S . Then $F_{(x,y)}$ is a flat $R_{(y)}$ -module if y_n is not a zerodivisor on $F_{(x,y)} \otimes_{R_{(y)}} S/I_{(z)}^*$.

Remark 1.11. Notice the weakness of assumptions on R . In particular, the above criterion allows one to verify flatness of modules over a much larger class of rings than that in [4, Thm. 4.1].

1.2. Plan of the paper. The rest of the paper is structured as follows: In Section 2, we recall the formalism of Hironaka's diagram of initial exponents as well as his criterion for flatness in the analytic category (Theorem 2.1). We follow there the excellent exposition of [7]. Theorem 2.1 is an essential component of the proofs of our main results, Theorems 1.2 and 1.3. The latter are proved in Section 3.

In Section 4, we prove a topological analogue of Theorem 1.3 – a criterion for (local) openness of a holomorphic mapping between complex-analytic spaces. Like our flatness criteria above, Theorem 4.1 is superior to the known effective openness criteria (see [1] and [3]) from the computational point of view. On the other hand, Theorem 4.1 requires an additional assumption that the source of the mapping be pure-dimensional. Example 4.4 proves the necessity of this assumption.

Finally, in the last section, we give the proofs of our algebraic criteria, Theorems 1.1 and 1.10. Roughly speaking, these follow from Theorems 1.2 and 1.3, respectively, by faithful flatness of completions of the rings of polynomials and the rings of convergent power series over the base ring. In Section 5, we also give an example of an explicit calculation of non-flatness, showing Theorem 1.1 at work.

2. HIRONAKA'S DIAGRAM OF INITIAL EXPONENTS AND FLATNESS CRITERION

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $R = \mathbb{K}\{y\}/I$ be a local analytic \mathbb{K} -algebra with the maximal ideal \mathfrak{m} , where $y = (y_1, \dots, y_n)$ and $I \subset \mathbb{K}\{y\}$ is a proper ideal. Let $x = (x_1, \dots, x_m)$ and define $R\{x\} := \mathbb{K}\{y, x\}/I \cdot \mathbb{K}\{y, x\}$. Given $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$ and a positive integer q , we will denote by x^β the monomial $x_1^{\beta_1} \dots x_m^{\beta_m}$, and by $x^{\beta,j}$ the q -tuple $(0, \dots, x^\beta, \dots, 0)$ with x^β in the j 'th place. Then, a q -tuple $G = (G_1, \dots, G_q) \in R\{x\}^q$ can be written as $G = \sum_{\beta,j} g_{\beta,j} x^{\beta,j}$, for some $g_{\beta,j} \in R$, where the indices (β, j) belong to $\mathbb{N}^m \times \{1, \dots, q\}$.

The mapping $R = \mathbb{K}\{y\}/I \rightarrow \mathbb{K}$, $g \mapsto g(0)$ of evaluation at zero, given by tensoring with $\otimes_R R/\mathfrak{m}$, induces the evaluation mapping

$$R\{x\}^q \rightarrow \mathbb{K}\{x\}^q, \quad G = \sum_{\beta,j} g_{\beta,j} x^{\beta,j} \mapsto G(0) = \sum_{\beta,j} g_{\beta,j}(0) x^{\beta,j}.$$

For a submodule M of $R\{x\}^q$, we will denote by $M(0)$ the image of M under the evaluation mapping.

Let L be any positive linear form on \mathbb{R}^m , $L(\beta) = \sum_{i=1}^m \lambda_i \beta_i$ ($\lambda_i > 0$). We define a total ordering of $\mathbb{N}^m \times \{1, \dots, q\}$ (denoted by L again) by lexicographic ordering of the $(m+2)$ -tuples $(L(\beta), j, \beta_1, \dots, \beta_m)$, where $\beta = (\beta_1, \dots, \beta_m)$ and

$(\beta, j) \in \mathbb{N}^m \times \{1, \dots, q\}$. For a q -tuple $G = \sum_{\beta, j} g_{\beta, j} x^{\beta, j} \in R\{x\}^q$, define the *support* of G as

$$\text{supp}(G) = \{(\beta, j) : g_{\beta, j} \neq 0\},$$

and the *initial exponent* of G (with respect to L) as

$$\exp_L(G) = \min_L \{(\beta, j) : (\beta, j) \in \text{supp}(G)\}.$$

Similarly, for the evaluated q -tuple $G(0)$, define

$$\text{supp}(G(0)) = \{(\beta, j) : g_{\beta, j}(0) \neq 0\},$$

and

$$\exp_L(G(0)) = \min_L \{(\beta, j) : (\beta, j) \in \text{supp}(G(0))\}.$$

Of course,

$$(2.1) \quad \text{supp}(G(0)) \subset \text{supp}(G).$$

We will also write $\text{supp}_x(G)$ and $\text{supp}_x(G(0))$ when necessary, to indicate relative to which variables the supports in question are.

For a submodule M of $R\{x\}^q$, the *diagram of initial exponents* of M (with respect to the total ordering L) is defined as

$$\mathfrak{N}_L(M) = \{\exp_L(G) : G \in M \setminus \{0\}\} \subset \mathbb{N}^m \times \{1, \dots, q\}.$$

Note that $\mathfrak{N}_L(M) + \mathbb{N}^m = \mathfrak{N}_L(M)$, since M is an $R\{x\}$ -module. (Indeed, $\exp_L(x^\gamma \cdot G) = \exp_L(G) + \gamma$ for any $G \in R\{x\}^q$ and $\gamma \in \mathbb{N}^m$, where $((\beta_1, \dots, \beta_m), j) + (\gamma_1, \dots, \gamma_m) = ((\beta_1 + \gamma_1, \dots, \beta_m + \gamma_m), j)$.)

The following flatness criterion of Hironaka (as adapted in [7]) lies at the heart of the proofs of our main theorems.

Theorem 2.1 (Hironaka, cf. [7, Thm. 7.9]). *If M is a submodule of $R\{x\}^q$ then there exists a positive linear form L on \mathbb{R}^m such that the following conditions are equivalent:*

- (i) $R\{x\}^q/M$ is a flat R -module.
- (ii) For every $G \in M$, $\text{supp}(G) \cap \mathfrak{N}_L(M(0)) = \emptyset$ implies that $G = 0$.

We will also often make use of the following simple observation.

Remark 2.2. If R is an integral domain, then any flat R -module is torsion-free over R . Indeed, this follows from the characterisation of flatness in terms of relations (see, e.g., [10, Cor. 6.5]).

3. PROOFS OF THE ANALYTIC FLATNESS CRITERIA

Proof of Theorem 1.2. Let $J = (y_1 - z_1 y_n, \dots, y_{n-1} - z_{n-1} y_n)$ and $\tilde{z} = (z_1, \dots, z_{n-1})$. Then, by definition of $\kappa : R \rightarrow S$, we have an isomorphism of R -modules

$$(3.1) \quad S \cong \mathbb{K}\{y, z\} / (y_1 - z_1 z_n, \dots, y_{n-1} - z_{n-1} z_n, y_n - z_n) \cong R\{\tilde{z}\} / J.$$

Choose $A = R\{x\}$, $x = (x_1, \dots, x_m)$, such that F is a finitely generated A -module. Then $F \cong A^q / M$ for some positive integer q and a submodule M of A^q . Hence

$$F \otimes_R S \cong R\{\tilde{z}, x\}^q / (J \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M).$$

Suppose first that F is flat over R . Since flatness is preserved by analytic base change ([16, § 6, Prop. 8]), it follows that $F \otimes_R S$ is flat and hence torsion-free over

S (by Remark 2.2). Hence also $F \tilde{\otimes}_R S$ is torsion-free over R , because R embeds into S , by (3.1).

Conversely, suppose that F is not R -flat. Then, by Theorem 2.1, one can choose a non-zero q -tuple $G = (G_1, \dots, G_q) \in M$ such that

$$(3.2) \quad \text{supp}_x(G) \subset (\mathbb{N}^m \times \{1, \dots, q\}) \setminus \mathfrak{N}_L(M(0))$$

(for some positive linear form L on \mathbb{R}^m). Since $G(0) \in M(0)$ and $\text{supp}_x(G(0)) \cap \mathfrak{N}_L(M(0)) = \emptyset$ (by (2.1)), it follows that $G(0) = 0$, that is, $G \in (y) \cdot A^q$. Define $G^\kappa = (G_1^\kappa, \dots, G_q^\kappa) \in R\{\tilde{z}, x\}^q$ by

$$G_j^\kappa(y, \tilde{z}, x) := G_j(z_1 y_n, \dots, z_{n-1} y_n, y_n, x), \quad j = 1, \dots, q.$$

Then $G^\kappa \in J \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M$, and $G^\kappa \in y_n \cdot R\{\tilde{z}, x\}^q$. Let $d \geq 1$ be the maximal integer for which $G^\kappa \in y_n^d \cdot R\{\tilde{z}, x\}^q$, and set $\tilde{G} := y_n^{-d} \cdot G^\kappa$. We claim that

$$\tilde{G} \in R\{\tilde{z}, x\}^q \setminus (J \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M).$$

We shall actually show a stronger statement, namely

$$(3.3) \quad \tilde{G}(0) = \tilde{G}(0, \tilde{z}, x) \notin (J \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M)(0).$$

Extend L to a positive linear form L' on $\mathbb{R}^{(n-1)+m}$ by setting $L'(\gamma, \beta) = \gamma_1 + \dots + \gamma_{n-1} + L(\beta)$, where $\gamma = (\gamma_1, \dots, \gamma_{n-1})$. Order $\mathbb{N}^{(n-1)+m} \times \{1, \dots, q\}$ relative to this form, and define $\mathfrak{N} \subset \mathbb{N}^{(n-1)+m} \times \{1, \dots, q\}$ to be the diagram of initial exponents (with respect to the variables \tilde{z} and x) of the evaluated ideal $(J \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M)(0)$. Since $J(0) = (0)$, and hence

$$(J \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M)(0) = (R\{\tilde{z}, x\} \cdot M)(0),$$

it follows that $(\gamma, \beta, j) \in \mathfrak{N}$ if and only if $(\beta, j) \in \mathfrak{N}_L(M(0))$, where $\gamma = (\gamma_1, \dots, \gamma_{n-1})$ is an exponent of the \tilde{z} -variables.

On the other hand, notice that $\text{supp}_{\tilde{z}, x}(\tilde{G}(0)) \neq \emptyset$, by the choice of d . Also, clearly, for every $(\gamma, \beta, j) \in \text{supp}_{\tilde{z}, x}(\tilde{G}(0))$, we have $(\beta, j) \in \text{supp}_x(G)$. Therefore, by (3.2), the non-empty support of $\tilde{G}(0)$ is contained in $\mathbb{N}^{(n-1)+m} \setminus \mathfrak{N}$, which proves (3.3).

Now, the class of \tilde{G} in $R\{\tilde{z}, x\}^q$ modulo $J \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M$ is a non-zero element of $F \tilde{\otimes}_R S$. But this element is a torsion element over R , because $y_n^d \cdot \tilde{G} = G^\kappa = 0$ in $F \tilde{\otimes}_R S$. Consequently, y_n^d (hence also y_n) is a zerodivisor on $F \tilde{\otimes}_R S$. \square

The following lemma will be used in the proof of Theorem 1.3, below.

Lemma 3.1. *Let $h(y) = \sum_{|\alpha|=d} h_\alpha y^\alpha \in \mathbb{K}[y_1, \dots, y_n]$ be a homogenous polynomial of degree d , where $d \geq 2$. There exist nonzero constants c_1, \dots, c_n such that, after a linear change of coordinates $y_j \mapsto y_j + c_j y_n$, $j = 1, \dots, n-1$, $y_n \mapsto c_n y_n$, the homogenous polynomial $h(y)$ contains a term $c \cdot y_n^d$ for some nonzero $c \in \mathbb{K}$.*

Proof. Set $E_h := \{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n : h_\alpha \neq 0\}$ and let $D := |E_h|$. If $D = 1$ then the lemma holds with $c_1 = \dots = c_n = 1$. Suppose then that $D \geq 2$. Let α^* be the maximal element of E_h with respect to the lexicographic ordering of the n -tuples $(\alpha_1, \dots, \alpha_n)$ in \mathbb{N}^n . Set $M := \max\{|h_\alpha|/|h_{\alpha^*}| : \alpha \in E_h\}$; then $M \geq 1$. Define

$$c_1 = (DM)^{d^{2(n-1)}}, \quad c_2 = (DM)^{d^{2(n-2)}}, \quad \dots, \quad c_{n-1} = (DM)^{d^2}, \quad c_n = DM,$$

and, for $\alpha = (\alpha_1, \dots, \alpha_n)$, set $p(\alpha) := d^{2(n-1)}\alpha_1 + d^{2(n-2)}\alpha_2 + \dots + d^2\alpha_{n-1} + \alpha_n$.

Now, after the substitution $y_j \mapsto y_j + c_j y_n$ ($1 \leq j \leq n-1$), $y_n \mapsto c_n y_n$, every term $h_\alpha y^\alpha$ of h gets transformed into a finite sum of terms, of which precisely one depends only on the variable y_n . This term is of the form

$$h_\alpha c_1^{\alpha_1} c_2^{\alpha_2} \dots c_n^{\alpha_n} y_n^d, \quad \text{that is,} \quad h_\alpha D^{p(\alpha)} M^{p(\alpha)} y_n^d.$$

Hence, $h(y)$ contains a term $c \cdot y_n^d$, where $c = \sum_{\alpha \in E_h} h_\alpha D^{p(\alpha)} M^{p(\alpha)}$. Therefore, to prove the lemma (i.e., to prove that $c \neq 0$) it suffices to show that

$$|h_{\alpha^*} D^{p(\alpha^*)} M^{p(\alpha^*)}| > \sum_{\alpha \in E_h \setminus \{\alpha^*\}} |h_\alpha D^{p(\alpha)} M^{p(\alpha)}|.$$

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in E_h \setminus \{\alpha^*\}$, there exists $j_0 < n$ such that $\alpha_j = \alpha_j^*$ for all $j \leq j_0$ and $\alpha_{j_0+1} \leq \alpha_{j_0+1}^* - 1$. Note that, since $|\alpha| = |\alpha^*|$ and α^* is the unique maximal element of E_h , we actually have $j_0 \leq n-2$. It follows that

$$\begin{aligned} p(\alpha) &\leq d^{2(n-1)} \alpha_1^* + \dots + d^{2(n-j_0)} \alpha_{j_0}^* + d^{2(n-j_0-1)} (\alpha_{j_0+1}^* - 1) + d^{2(n-j_0-2)} d \\ &= d^{2(n-1)} \alpha_1^* + \dots + d^{2(n-j_0)} \alpha_{j_0}^* + d^{2(n-j_0-1)} \alpha_{j_0+1}^* - d^{2n-2j_0-3} (d-1) \\ &\leq p(\alpha^*) - d(d-1). \end{aligned}$$

Hence, for every $\alpha \in E_h \setminus \{\alpha^*\}$,

$$|h_\alpha D^{p(\alpha)} M^{p(\alpha)}| = \frac{|h_\alpha|}{|h_{\alpha^*}|} \cdot |h_{\alpha^*}| D^{p(\alpha)} M^{p(\alpha)} \leq M |h_{\alpha^*}| (DM)^{p(\alpha^*)-d(d-1)}.$$

Consequently

$$\begin{aligned} \sum_{\alpha \in E_h \setminus \{\alpha^*\}} |h_\alpha D^{p(\alpha)} M^{p(\alpha)}| &\leq DM |h_{\alpha^*}| (DM)^{p(\alpha^*)-d(d-1)} \\ &= |h_{\alpha^*}| (DM)^{p(\alpha^*)-d(d-1)+1} < |h_{\alpha^*}| (DM)^{p(\alpha^*)}, \end{aligned}$$

because $d(d-1) \geq 2$, as $d \geq 2$. \square

Remark 3.2. Note that, in the coordinate change of Lemma 3.1, one can actually require that $c_n = 1$, i.e., that the variable y_n stays unaffected. Indeed, given a change with non-zero coefficients c_1, \dots, c_n as above, simply change the coordinates once more, by setting $y_j \mapsto y_j$ for $j = 1, \dots, n-1$, and $y_n \mapsto 1/c_n \cdot y_n$. Then the term $c \cdot y_n^d$ of h gets transformed to $c/c_n^d \cdot y_n^d$ with $c/c_n^d \neq 0$.

Proof of Theorem 1.3. As in the proof of Theorem 1.2, set $\tilde{z} = (z_1, \dots, z_{n-1})$ and $J = (y_1 - z_1 y_n, \dots, y_{n-1} - z_{n-1} y_n)$. The composite of the canonical map $R = \mathbb{K}\{y\}/I \rightarrow S/(\kappa(I))$ (induced by κ) and the epimorphism $S/(\kappa(I)) \rightarrow S/I^*$ makes S/I^* into an R -module. Let J^* be the quotient modulo $I \cdot \mathbb{K}\{y, \tilde{z}\}$ of the ideal

$$\{g : y_n^k \cdot g \in J + I \cdot \mathbb{K}\{y, \tilde{z}\} \text{ for some } k \in \mathbb{N}\}$$

in $\mathbb{K}\{y, \tilde{z}\}$. Then we have an isomorphism of R -modules

$$(3.4) \quad S/I^* \cong R\{\tilde{z}\}/J^*.$$

Choose $A = R\{x\}$, $x = (x_1, \dots, x_m)$, such that F is a finitely generated A -module. Then $F \cong A^q/M$ for some positive integer q and a submodule M of A^q . Hence

$$F \otimes_R S/I^* \cong R\{\tilde{z}, x\}^q / (J^* \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M).$$

Suppose first that F is flat over R , and S/I^* is an integral domain. Since flatness is preserved by analytic base change ([16, § 6, Prop. 8]), it follows that $F \tilde{\otimes}_R S/I^*$ is flat and hence torsion-free over S/I^* (by Remark 2.2). If, moreover, R embeds into S/I^* , then $F \tilde{\otimes}_R S/I^*$ is also torsion-free as an R -module. This proves claim (ii) of the theorem.

Conversely, suppose that F is not flat over R . Then, by Theorem 2.1, one can choose a non-zero q -tuple $G = (G_1, \dots, G_q) \in M$ such that

$$(3.5) \quad \text{supp}_x(G) \subset (\mathbb{N}^m \times \{1, \dots, q\}) \setminus \mathfrak{N}_L(M(0))$$

(for some positive linear form L on \mathbb{R}^m). Since $G(0) \in M(0)$ and $\text{supp}_x(G(0)) \cap \mathfrak{N}_L(M(0)) = \emptyset$ (by (2.1)), it follows that $G(0) = 0$, that is, $G \in \mathfrak{m} \cdot A^q$, where \mathfrak{m} denotes the maximal ideal of R .

Write $G = \sum_{\beta, j} g_{\beta, j} x^{\beta, j}$, and set

$$(3.6) \quad d := \max\{k \in \mathbb{N} : g_{\beta, j} \in \mathfrak{m}^k \text{ for all } (\beta, j) \in \text{supp}_x(G)\}.$$

Choose $(\beta^*, j^*) \in \mathbb{N}^m \times \{1, \dots, q\}$ such that $g_{\beta^*, j^*} \in \mathfrak{m}^d \setminus \mathfrak{m}^{d+1}$, and write $g_{\beta^*, j^*}(y) = \sum_{\nu \geq d} g'_{\beta^*, j^*}(\nu)(y)$, where each $g'_{\beta^*, j^*}(\nu)$ is a homogenous polynomial of degree ν . Now, by Lemma 3.1, after a linear change of the y -variables (which does not affect y_n , by Remark 3.2), we can assume that the initial form g_{β^*, j^*}^d of g_{β^*, j^*} contains a term $c^* \cdot y_n^d$ for some $c^* \neq 0$ (here, we identify y_n with its class in \mathfrak{m} ; this can be done, because no power of y_n belongs to \mathfrak{m} , for else we would have $I^* = S$, contrary to our assumption).

Next, we proceed as in the proof of Theorem 1.2. Define $G^\kappa = (G_1^\kappa, \dots, G_q^\kappa) \in R\{\tilde{z}, x\}^q$ by

$$G_j^\kappa(y, \tilde{z}, x) := G_j(z_1 y_n, \dots, z_{n-1} y_n, y_n, x), \quad j = 1, \dots, q.$$

Then $G^\kappa \in J^* \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M$, and $G^\kappa \in y_n \cdot R\{\tilde{z}, x\}^q$. Notice that, by (3.6), d is the maximal integer for which $G^\kappa \in y_n^d \cdot R\{\tilde{z}, x\}^q$. Set $\tilde{G} := y_n^{-d} \cdot G^\kappa$. We claim that

$$\tilde{G} \in R\{\tilde{z}, x\}^q \setminus (J^* \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M).$$

We shall actually show a stronger statement, namely

$$(3.7) \quad \tilde{G}(0, 0) = \tilde{G}(0, 0, x) \notin (J^* \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M)(0, 0),$$

where we evaluate at zero both the y and \tilde{z} -variables.

Let $\tilde{\mathfrak{N}} \subset \mathbb{N}^m \times \{1, \dots, q\}$ denote the diagram of initial exponents (with respect to the variables x) of the evaluated ideal $(J^* \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M)(0, 0)$. By assumption, I^* is a proper ideal in S . Hence $J^*(0, 0) = (0)$, and so $\tilde{\mathfrak{N}} = \mathfrak{N}_L(M(0))$.

On the other hand, $(\beta^*, j^*) \in \text{supp}_x(\tilde{G}(0, 0))$, because

$$y_n^{-d} \cdot g_{\beta^*, j^*}(z_1 y_n, \dots, z_{n-1} y_n, y_n)|_{y=\tilde{z}=0} = c^* \neq 0.$$

Thus $\text{supp}_x(\tilde{G}(0, 0)) \neq \emptyset$. Also, clearly, $\text{supp}_x(\tilde{G}(0, 0)) \subset \text{supp}_x(G)$. Therefore, by (3.5), the non-empty support of $\tilde{G}(0, 0)$ is contained in $\mathbb{N}^m \setminus \tilde{\mathfrak{N}}$, which proves (3.7).

Now, the class of \tilde{G} in $R\{\tilde{z}, x\}^q$ modulo $J^* \cdot R\{\tilde{z}, x\}^q + R\{\tilde{z}, x\} \cdot M$ is a non-zero element of $F \tilde{\otimes}_R S/I^*$. But this element is a torsion element over R , because $y_n^d \cdot \tilde{G} = G^\kappa = 0$ in $F \tilde{\otimes}_R S/I^*$. Consequently, y_n^d (hence also y_n) is a zerodivisor on $F \tilde{\otimes}_R S/I^*$, which proves claim (i) of the theorem. \square

4. OPENNESS CRITERION

Let $\varphi : X \rightarrow Y$ be a morphism of complex-analytic subspaces of \mathbb{C}^m and \mathbb{C}^n respectively. Assume that X is of pure dimension, Y is positive-dimensional and locally irreducible, and $\varphi(0) = 0$. Let $\sigma : \mathbb{C}^n \rightarrow \mathbb{C}^n$ denote the blowing-up of the origin restricted to a local coordinate chart in which the strict transform Z of Y passes through the origin (cf. Remark 1.9). The following is a topological analogue of Theorem 1.8.

Theorem 4.1. *The map φ is open at $0 \in X$ if and only if its pull-back $X \times_Y Z \rightarrow Z$ has no isolated algebraic vertical component at $(0,0)$. (Equivalently, the induced map $X \times_Y Z \rightarrow Y$ has no isolated algebraic vertical component at $(0,0)$.)*

Here, by a map φ open at a point ξ we mean a map whose restriction to a certain open neighbourhood of ξ is an open map. In other words, φ_ξ is a germ of an open map.

Remark 4.2. Note that in the category of continuous maps between topological spaces (hence also in the category of holomorphic maps between complex-analytic spaces), openness implies universal openness. That is, if $\varphi_\xi : X_\xi \rightarrow Y_\eta$ is an open germ and $\psi_\zeta : Z_\zeta \rightarrow Y_\eta$ is an arbitrary map-germ, then the pull-back of φ_ξ by ψ_ζ , $(X \times_Y Z)_{(\xi,\zeta)} \rightarrow Z_\zeta$ is again open.

Proof of Theorem 4.1. If φ_0 is open, then its pull-back by $\sigma_0|_{Z_0}$, $(X \times_Y Z)_{(0,0)} \rightarrow Z_0$ is also open, by Remark 4.2. Therefore, $(X \times_Y Z)_{(0,0)}$ has no isolated algebraic vertical component over Z_0 , hence also over Y_0 (because $\sigma|_Z : Z \rightarrow Y$ is dominant).

Conversely, suppose that φ_0 is not open. Then, by the Remmert Open Mapping Theorem (see [17, §V.6, Thm. 2]), we have $\text{fbd}_0 \varphi > \dim X - \dim_0 Y$, or

$$(4.1) \quad \dim X \leq \dim_0 Y - 1 + \text{fbd}_0 \varphi,$$

where $\text{fbd}_\xi \varphi$ denotes the fibre dimension of φ at a point ξ , $\dim_\xi \varphi^{-1}(\varphi(\xi))$. Since $\sigma|_Z$ is a biholomorphism outside $\sigma^{-1}(0)$, we can write $X \times_Y Z = T \cup T'$, where $T' = (\sigma|_Z)^{-1}(0) \times \varphi^{-1}(0)$ and T is biholomorphic with $\varphi^{-1}(Y \setminus \{y_n = 0\})$. One readily sees that $\dim T \leq \dim X$ and $\dim_0(\sigma|_Z)^{-1}(0) = \dim_0 Y - 1$. Therefore, by (4.1),

$$\dim T \leq \dim_0 Y - 1 + \text{fbd}_0 \varphi = \dim_{(0,0)} T'.$$

It follows that $\dim_{(0,0)} T' = \dim_{(0,0)}(X \times_Y Z)$, and hence T' must contain an isolated irreducible component of $X \times_Y Z$ through $(0,0)$. By definition of T' , such a component is mapped into $(\sigma|_Z)^{-1}(0)$ in Z , and so it is algebraic vertical (over Z_0 , as well as over Y_0). \square

In the algebraic setting, Theorem 4.1 has the following analogue over an arbitrary field \underline{k} (cf. [3, Thm. 1.1]):

Theorem 4.3. *Let Y be a scheme of finite type over a field \underline{k} , and let $\varphi : X \rightarrow Y$ be a morphism which is locally of finite type. Assume that Y is normal and positive-dimensional and X is of pure dimension. Let \mathfrak{m} be a closed point of Y , and let $\sigma : Z \rightarrow Y$ denote the blowing-up of Y at \mathfrak{m} . Then φ is open at a point $\mathfrak{p} \in \varphi^{-1}(\mathfrak{m})$ if and only if the pullback of φ by σ , $X \times_Y Z \rightarrow Z$ has no vertical irreducible components.*

Openness of φ at \mathfrak{p} means, as above, openness in some neighbourhood of \mathfrak{p} . A *vertical* irreducible component is (by analogy with the local analytic case) an irreducible component of the source whose image is nowhere-dense in the target.

The proof of Theorem 4.3 is virtually identical with the one above, because: (a) openness of a map with a normal target is equivalent to universal openness (see [14, Cor. 14.4.3]), and (b) a map $\varphi : X \rightarrow Y$ with X pure-dimensional and Y normal is open if and only if φ is dominating and the fibres of φ are equidimensional and of constant dimension (see [14, Cor. 14.4.6]).

Remark 4.4. Interestingly, Theorems 4.1 and 4.3 are false, in general, without the pure-dimensionality assumption on X . This can be seen in the following example.

Example 4.5. Let $X = X_1 \cup X_2$ be a subset of \mathbb{C}^9 (with coordinates $(t, x) = (t_1, t_2, t_3, x_1, \dots, x_6)$), where

$$\begin{aligned} X_1 &= \{(t, x) : t_1x_1 + t_2x_2 + t_3x_3 = t_2x_1 + t_1x_2 = x_4 = x_5 = x_6 = 0\}, \\ X_2 &= \{(t, x) : t_1 = t_2 = t_3 = 0\}. \end{aligned}$$

Clearly, X_2 is irreducible, of dimension 6. We claim that X_1 is of pure dimension 4. To see this, set $A = \{(t, x) \in X_1 : \det \begin{bmatrix} t_1 & t_2 \\ t_2 & t_1 \end{bmatrix} = 0\}$. In $X_1 \setminus A$, one can solve the first two defining equations of X_1 for x_1 and x_2 , hence $X_1 \setminus A$ is a 4-dimensional manifold. On the other hand, it is not difficult to see that $\dim A = 3$. Since X_1 is defined by 5 equations in \mathbb{C}^9 , it follows that $\dim_\xi X_1 \geq 4$ for every $\xi \in X_1$. Therefore A is nowhere-dense in X_1 and $X_1 = \overline{X_1} \setminus A$ is of pure dimension 4.

Define $\varphi : X \rightarrow Y = \mathbb{C}^3$ as

$$(t, x) \mapsto (t_1 + x_4, t_2 + x_5, t_3 + x_6).$$

We claim that φ is not open at 0. For this, it suffices to show that $\varphi|_{X_1}$ is not open in any neighbourhood of 0. Consider the set $W = \{(t, x) \in X_1 : t_3 = 0, t_1 = t_2 \neq 0\}$. Then $W \subset X_1 \setminus X_2$, and for every $\xi \in W$, we have $\text{fbd}_\xi \varphi = 2$. On the other hand, the generic fibre dimension of $\varphi|_{X_1}$ is 1, as is easy to see. Therefore $\varphi|_{X_1}$ is not open at any such ξ , by the Remmert Open Mapping Theorem. But W is adherent to $0 \in \mathbb{C}^9$, which proves our claim.

Finally, let $\sigma : \mathbb{C}^3 \rightarrow Y$ be given as $\sigma(z_1, z_2, z_3) = (z_1z_3, z_2z_3, z_3)$. We shall show that $\varphi' : X \times_Y Z \rightarrow Z$, the pullback of φ by σ has no isolated vertical components through $(0, 0)$. As in the proof of Theorem 4.1, write $X \times_Y Z = T \cup T'$, where $T' = \varphi'^{-1}(0)$ and T is biholomorphic with $X \setminus \varphi^{-1}(\{y_3 = 0\})$. Since φ_0 has no isolated algebraic vertical components itself, it follows that the image of (an arbitrarily small neighbourhood of a point ξ near 0 in) T under φ' contains an open subset of Y . Therefore, if φ' has an isolated algebraic vertical component Σ through $(0, 0)$, then $\Sigma \subset T'$. But T' is a fibre of an open map ψ' defined as the pull-back by σ of $\psi := \varphi|_{X_2}$. By Theorem 4.1, ψ' has no isolated algebraic vertical components, which proves that there is no such Σ .

5. ALGEBRAIC CASE

Our flatness criterion in the algebraic setting can be reduced to the analytic case, settled above, by means of the following simple but fundamental observation.

Remark 5.1. Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Suppose that M is a module over the local ring $R = \mathbb{K}[x]_{(x)}$, where $x = (x_1, \dots, x_n)$. Then

- (i) M is a flat R -module if and only if $M \cdot \mathbb{K}\{x\}$ is a flat $\mathbb{K}\{x\}$ -module.
- (ii) Given non-zero $r \in R$, r is a zerodivisor on M if and only if r is a zerodivisor on $M \cdot \mathbb{K}\{x\}$.

Indeed, the modules M and $M \cdot \mathbb{K}\{x\}$ share the same completion \widehat{M} (with respect to the (x) -adic topology in R and $\mathbb{K}\{x\}$, respectively) over $\mathbb{K}[[x]] = \widehat{R} = \widehat{\mathbb{K}\{x\}}$. By faithful flatness of \widehat{R} over R ([8, Ch. III, § 3, Prop. 6] and [8, Ch. III, § 5, Prop. 9]), M is R -flat if and only if \widehat{M} is $\mathbb{K}[[x]]$ -flat. By faithful flatness of $\widehat{\mathbb{K}\{x\}}$ over $\mathbb{K}\{x\}$, in turn, \widehat{M} is $\mathbb{K}[[x]]$ -flat if and only if $M \cdot \mathbb{K}\{x\}$ is $\mathbb{K}\{x\}$ -flat.

Claim (ii) follows from faithful flatness of completion applied to the sequence $0 \rightarrow M \xrightarrow{\cdot x} M$.

Proof of Theorem 1.1. Let \underline{k} , $R = \underline{k}[y_1, \dots, y_n]$, and $F \cong R[x]^q/M$ be as in the statement of the theorem. Let $S = \mathbb{K}[z_1, \dots, z_n]$ and let $\kappa : R \rightarrow S$ be the morphism

$$\kappa(y_1) = z_1 z_n, \dots, \kappa(y_{n-1}) = z_{n-1} z_n, \kappa(y_n) = z_n.$$

Then, the condition $\widetilde{M} = \widetilde{M} : y_n$ is equivalent to saying that y_n is not a zerodivisor in $F_{(x,y)} \otimes_{R_{(y)}} S_{(z)}$.

Suppose first that $F_{(x,y)}$ is a flat $R_{(y)}$ -module. Since flatness is preserved by base change, it follows that $F_{(x,y)} \otimes_{R_{(y)}} S_{(z)}$ is flat and hence torsion-free over $S_{(z)}$ (by Remark 2.2). Hence also $F_{(x,y)} \otimes_{R_{(y)}} S_{(z)}$ is torsion-free over $R_{(y)}$, because $R_{(y)}$ embeds into $S_{(z)}$. In particular, y_n is not a zerodivisor on $F_{(x,y)} \otimes_{R_{(y)}} S_{(z)}$.

Conversely, suppose that $F_{(x,y)}$ is not flat over $R_{(y)}$. We will proceed in three steps, depending on \underline{k} . First, suppose that $\underline{k} = \mathbb{C}$. Then y_n is a zerodivisor on $F_{(x,y)} \otimes_{R_{(y)}} S_{(z)}$, by Theorem 1.2 and Remark 5.1.

Next, suppose that \underline{k} is algebraically closed. Then our result follows from the above case, by the Tarski-Lefschetz Principle (see, e.g., [18]), as flatness can be expressed in terms of a finite number of relations ([10, Cor. 6.5]).

Finally, let \underline{k} be an arbitrary field of characteristic zero, and let \mathbb{K} denote an algebraic closure of \underline{k} . Set $R' := R \otimes_{\underline{k}} \mathbb{K}$, $S' := S \otimes_{\underline{k}} \mathbb{K}$, and $F' := F \otimes_{\underline{k}} \mathbb{K}$. It is not difficult to verify that R' is a faithfully flat R -module (see, e.g., [3]). Therefore, $F_{(x,y)}$ is not $R_{(y)}$ -flat if and only if $F'_{(x,y)}$ is not $R'_{(y)}$ -flat. By the previous part of the proof, the latter implies that y_n is a zerodivisor on $F'_{(x,y)} \otimes_{R'_{(y)}} S'_{(z)}$. But $F'_{(x,y)} \otimes_{R'_{(y)}} S'_{(z)} \cong (F_{(x,y)} \otimes_{R_{(y)}} S_{(z)}) \otimes_{R_{(y)}} R'_{(y)}$, so y_n is also a zerodivisor on $F_{(x,y)} \otimes_{R_{(y)}} S_{(z)}$, which completes the proof. \square

Proof of Theorem 1.10. Let $R = \mathbb{K}[y_1, \dots, y_n]/I$, where I is a proper ideal in $\mathbb{K}[y_1, \dots, y_n]$. Let $A = R[x_1, \dots, x_m]/Q$ be an R -algebra of finite type, and let F be a finitely generated A -module. Suppose that $F_{(x,y)}$ is not flat over $R_{(y)}$.

Let $\varphi : X \rightarrow Y$ be the \mathbb{K} -analytic mapping of \mathbb{K} -analytic spaces associated to the morphism $\text{Spec } A \rightarrow \text{Spec } R$, and let \widetilde{F} denote the finite $\mathcal{O}_{X,0}$ -module $F_{(x,y)} \cdot \mathcal{O}_{X,0}$. The problem being local, we can assume that Y is a subspace of \mathbb{K}^n . Let further $\sigma : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be the mapping sending $(z_1, \dots, z_{n-1}, z_n)$ to $(z_1 z_n, \dots, z_{n-1} z_n, z_n)$, so that the pull-back homomorphism $\sigma_0^* : \mathbb{K}\{y\} \rightarrow \mathbb{K}\{z\}$ is given by the same formulas as κ in the statement of the theorem. By assumption on the ideal I^* , the strict transform Z of Y (under σ) passes through the origin (in \mathbb{K}^n with the z -variables).

Now, \tilde{F} is a non-flat $\mathcal{O}_{Y,0}$ -module, by Remark 5.1. Hence, by Theorem 1.8, y_n is a zerodivisor on $\tilde{F} \otimes_{\mathcal{O}_{Y,0}} \mathcal{O}_{Z,0}$. Thus, by Remark 5.1 again, y_n is a zerodivisor on $F_{(x,y)} \otimes_{R_{(y)}} S/I_{(z)}^*$, as required. \square

Example 5.2. Consider the polynomial mapping $\varphi : X \rightarrow Y = \mathbb{C}^3$ from Example 4.5. That is, let $X = X_1 \cup X_2$ be a subset of \mathbb{C}^9 (with coordinates $(t, x) = (t_1, t_2, t_3, x_1, \dots, x_6)$), where

$$X_1 = \{(t, x) : t_1x_1 + t_2x_2 + t_3x_3 = t_2x_1 + t_1x_2 = x_4 = x_5 = x_6 = 0\},$$

$$X_2 = \{(t, x) : t_1 = t_2 = t_3 = 0\},$$

and let

$$\varphi(t, x) = (t_1 + x_4, t_2 + x_5, t_3 + x_6).$$

By Example 4.5, φ is not open at 0. Since flatness implies openness, by a theorem of Douady ([9]), it follows that φ is not flat at 0. This can be verified directly, by means of Theorem 1.1, as follows:

X can be embedded into $\mathbb{C}^9 \times Y$ via the graph of φ . Hence, the coordinate ring $A[X]$ of X can be identified with $\mathbb{C}[y, t, x]/(I_1 + I_2)$, where

$$I_1 = (y_1 - t_1 - x_4, y_2 - t_2 - x_5, y_3 - t_3 - x_6) \quad \text{and}$$

$$I_2 = (t_1x_1 + t_2x_2 + t_3x_3, t_2x_1 + t_1x_2, x_4, x_5, x_6) \cap (t_1, t_2, t_3).$$

Set $F = A[X]$ and $R = \mathbb{C}[y]$. We want to prove that $F_{(t,x,y)}$ is not flat over $R_{(y)}$.

Let \tilde{I}_1 (resp. \tilde{I}_2) denote the ideal obtained from I_1 (resp. I_2) by substituting y_1y_3 for y_1 , and y_2y_3 for y_2 ; i.e.,

$$\tilde{I}_1 = (y_1y_3 - t_1 - x_4, y_2y_3 - t_2 - x_5, y_3 - t_3 - x_6) \quad \text{and}$$

$$\tilde{I}_2 = (t_1x_1 + t_2x_2 + t_3x_3, t_2x_1 + t_1x_2, x_4, x_5, x_6) \cap (t_1, t_2, t_3).$$

By Theorem 1.1, the non-flatness of $F_{(t,x,y)}$ over $R_{(y)}$ can be detected by showing that $(\tilde{I}_1 + \tilde{I}_2) : y_n \not\supseteq \tilde{I}_1 + \tilde{I}_2$. We have verified, with help of a computer algebra system Singular (see, e.g., [13]), that $(\tilde{I}_1 + \tilde{I}_2) : y_n$ contains an element $x_6y_2 - x_5$, which does not belong to $\tilde{I}_1 + \tilde{I}_2$. This is easy to see. Indeed, on the one hand,

$$\begin{aligned} y_3(x_6y_2 - x_5) &= x_6y_2y_3 - x_5y_3 \equiv_{\tilde{I}_1} x_6(t_2 + x_5) - x_5y_3 = t_2x_6 + x_5x_6 - x_5y_3 \\ &\equiv_{\tilde{I}_2} 0 + x_5x_6 - x_5y_3 \equiv_{\tilde{I}_1} x_5x_6 - x_5(t_3 + x_6) = -t_3x_5 \equiv_{\tilde{I}_2} 0. \end{aligned}$$

On the other hand, suppose that $x_6y_2 - x_5 \in \tilde{I}_1 + \tilde{I}_2$. Then, after evaluating at zero the variables $y_1, y_2, y_3, t_1, t_3, x_1, x_2, x_3, x_4$, and x_6 , we would get

$$-x_5 \in (t_2 + x_5, t_2x_5) \cdot \mathbb{C}[t_2, x_5],$$

which is false.

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